The probabilitiy distribution density of random values of squared functional on Wiener process trajectories

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1. We consider the problem about the distribution density calculation of the random variable $J_T[w]$ in this work. Here

$$J_T[x] \ = \ \int_0^T x^2(t)dt \tag{1}$$

is the functional in the $\mathbf{L}_2([0,T])$ space and $\{w(t); t \in [0,T]\}$, $T > 0$ are trajectories of the standard Wiener process on $[0,T]$, i.e. $\mathsf{E}w^2(t) = t$. This problem is classical as well as analogous problems concerns the probability distribution density calculation of random variables performed by additive squared functionals on trajectories of gaussian random processes. For such processes, the calculation problem connected with characteristic functions of random values under consideration is solved principally. Up to present time, a great number specific problems are solved on the basis of this method which have various applications (see [1], [3]). For example, such a result concerns the normal markovian process (the Ornstein-Uhlenbeck process) has been still obtained in the work [2]. However, the problem of restoration of the distribution density on the basis of obtained characteristic functions remains weakly investigated in the sense of the constructing of approximations with the guaranteed accuracy being suitable for use whereas in the area of the random variable changing. The usual approach to the solution of this problem (see, for example, [4]) results in some approximated formulas for distribution densities suitable for the estimation of great fluctuation probabilities, i.e. in the asymptotic area $x \to \infty$ of the random value changing. Here, we study the calculation problem of successive approximations of the probability distribution density $f(x)$ connected with random values of the functional (1) defined in any compact interval $[0, M]$, $M > 0$.

2. Since trajectories $\{w(t); t \geq 0\}$ of the standard Wiener process $\mathsf{E}w^2(t) = t$ are continuous with the probability one, then the random value $J_T[w]$ is determined almost sure for each of them.

The generating function of the random value (1) is evaluated by the for-

mula (see, for example, [1])

$$Q_T(\lambda) \ = \ \mathsf{E}\exp(-\lambda J_T[w]) \ = \ \left[\operatorname{ch}\left(\lambda^{1/2}T\right)\right]^{-1/2}\,, \quad Q_T(\lambda) = Q_1(\lambda T^2)\,.$$

The distribution density $f(x)$ of the random value $J_T[w]$ is defined by the inverse Laplace transformation

$$f(x) \ = \ \frac{1}{2\pi i}\int\limits_{-i\infty+c}^{i\infty+c} e^{\lambda x}\left[\operatorname{ch}\left(\lambda^{1/2}T\right)\right]^{-1/2} d\lambda\,, \tag{2}$$

where c >0 and, for the integrated function, the cross-cut in the complex plane λ is done along the negative part of the real axe.

It is convenient to introduce the density $g(x) = T^2 f(T^2 x)$. The replacement of the integration variable in Eq.(2) gives

$$g(x) \ = \ \frac{1}{2\pi i}\int\limits_{-i\infty+c}^{i\infty+c} e^{\lambda x}\left[\operatorname{ch}\left(\lambda^{1/2}\right)\right]^{-1/2} d\lambda \ =$$

$$= \ \frac{1}{\sqrt{2}\pi i}\int\limits_{-i\infty+c}^{i\infty+c}\left(\frac{\exp(2\lambda x - \lambda^{1/2})}{1+\exp(-2\lambda^{-1/2})}\right)^{1/2} d\lambda \tag{3}$$

in this case. Let us prove the following theorem.

T h e o r e m. *The density $g(x)$ is represented by the following absolutely converging series*

$$g(x) \ = \ \sqrt{\frac{2}{\pi x^3}}\sum_{l=0}^{\infty}(-1)^l\,\frac{(2l)!}{4^l(l!)^2}\,(l+1/4)\exp\left(-\frac{(l+1/4)^2}{x}\right)\,, \tag{4}$$

the Nth remainder of this series is estimated by the value

$$|g(x) - g_{N-1}(x)| \le \sqrt{\frac{2}{\pi x^3}}\left(\frac{(2N)!}{4^N(N!)^2}\right)(N+1/4)\exp\left(-\frac{(N+1/4)^2}{x}\right)\,. \tag{5}$$

□ We shall put $c = 0$ in Eq.(3) since singularities of integrated function are on the negative part of the real axe. We deform the integration contour to the contour C consisting of the consecutive transitions of following ways

$\{s - i\varepsilon; s \in (-\infty; 0]\}$, $\{\varepsilon e^{is}; s \in [-\pi/2; \pi/2]\}$, $\{-s + i\varepsilon; s \in [0; +\infty)\}$. Such a deformation is permissible since

$$\left|\mathrm{ch}\left(\lambda^{1/2}\right)\right|^2 = \mathrm{ch}\left(\lambda^{1/2}\right)\mathrm{ch}\left((\lambda^*)^{1/2}\right) = \frac{1}{2}\left[\mathrm{ch}\left(2\mathrm{Re}(\lambda^{1/2})\right) + \mathrm{ch}\left(2i\mathrm{Im}(\lambda^{1/2})\right)\right] >$$

$$> \frac{1}{2}\left(\mathrm{ch}\left(2R^{1/2}\cos(\varphi/2)\right) - 1\right) = \mathrm{sh}^2\left(R^{1/2}\cos(\varphi/2)\right),$$

where $\lambda = Re^{i\varphi}$ and, on the arch $\{\lambda; \varphi \in [\pi/2; \pi)\}$ of the circle, the following estimation of the integrated expression module in Eq.(3) is valid

$$\left|\frac{\exp(\lambda x)}{\left(\mathrm{ch}(\lambda^{1/2})\right)^{1/2}}\right| \leq \frac{\exp(xR\cos\varphi)}{\left[\mathrm{sh}(R^{1/2}\cos(\varphi/2))\right]^{1/2}}.$$

It guarantees the fulfillment of the Jordan condition at $x > 0$ on the arch $R^{1/2}\cos(\varphi/2) < \varepsilon$ at any small $\varepsilon > 0$ since $\cos\varphi < 0$. The same takes place for the arch $\{\lambda; \varphi \in (-\pi, \pi/2]\}$.

In Eq.(3) we shall realize the replacement of the integration variable $\lambda^{1/2} = q$, then $\lambda = q^2$, $d\lambda = 2qdq$. Thus, the contour C in the complex plane λ will be transformed to the line and, after the transition to the limit $\varepsilon \to 0$, it will be transformed to the line $\{q = is; s \in \mathbb{R}\}$ in the complex plane q. After these transformations, we have

$$g(x) = \frac{\sqrt{2}}{\pi i}\int_{-i\infty}^{i\infty} q\left(\frac{\exp(2q^2x - q)}{1 + \exp(-2q)}\right)^{1/2} dq.$$

Now, we pass to the integration on the variable $s, q = c + is, dq = ids$. Then we obtain

$$g(x) = i\frac{\sqrt{2}}{\pi}\int_{-\infty}^{+\infty} s\left(\frac{\exp(-2xs^2 - is)}{1 + \exp(-2is)}\right)^{1/2} ds. \qquad (6)$$

In the last integral, we shall produce the shift $s + i(4x)^{-1} \Rightarrow s$ of the integration variable. Therefore, we obtain

$$g(x) = i\frac{\sqrt{2}}{\pi}\int_{-\infty}^{+\infty} s\left(\frac{\exp(-2x(s - i(4x)^{-1})^2 - (8x)^{-1})}{1 + \exp(-2i(s + i(4x)^{-1}) - (2x)^{-1})}\right)^{1/2} ds =$$

$$= \frac{\sqrt{2}}{\pi}\exp(-(16x)^{-1})\int\limits_{-\infty}^{+\infty}(is+(4x)^{-1})\left(\frac{\exp(-2xs^2)}{1+e^{-1/(2x)}\exp(-2is)}\right)^{1/2}ds\,. \quad (7)$$

We decompose the denominator of the integrated expression in Eq.(7) into the series converged at any $x > 0$ and at any $s \in \mathbb{R}$

$$(1+e^{-1/(2x)}\exp(-2is))^{-1/2} = \sum_{l=0}^{\infty}(-1)^l\frac{(2l-1)!!}{2^l l!}\exp(-l/(2x))\exp(-2ils).$$

The convergence is uniform in any area $[0, M] \times \mathbb{R}$ in the plane (x, s), $M > 0$. Substituting the last expression in Eq.(7), we obtain

$$g(x) = \frac{\sqrt{2}}{\pi}\exp(-(16x)^{-1})\int\limits_{-\infty}^{+\infty}(is+(4x)^{-1})\exp(-xs^2)\ \times$$

$$\times\ \left[\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{4^l(l!)^2}\exp(-2ils)\exp(-l/(2x))\right]ds =$$

$$= \frac{\sqrt{2}}{\pi}\exp(-(16x)^{-1})\left[\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{4^l(l!)^2}\int\limits_{-\infty}^{+\infty}(is+(4x)^{-1})\exp(-xs^2)\ \times\right.$$

$$\left.\times\ \exp\left(-x(s+il/x)^2 - x^{-1}l(l+1/2)\right)\right]ds\,.$$

The transposition of summation and integration is based on the uniform convergence of the series on s at any fixed x.

In each summand of the sum, we shall produce the shift $s + il/x \Rightarrow s$ of the integration variable,

$$g(x) = \frac{\sqrt{2}}{\pi}\exp(-(16x)^{-1})\ \times$$

$$\times\int\limits_{-\infty}^{+\infty}\left[\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{4^l(l!)^2}\left(is+x^{-1}\left[l+1/4\right]\right)\exp\left(-xs^2-x^{-1}l(l+1/2)\right)\right]ds\,. \quad (8)$$

Let us represent the integral as the sum of two integrals in accordance with the expression in the bracket before exponent. The integral corresponding

the summand is converts to the zero

$$i\int\limits_{-\infty}^{+\infty} s\left[\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{4^l(l!)^2}\exp(-xs^2)\exp\left(-x^{-1}l(l+1/2)\right)\right]ds = 0\,,$$

due to the oddness of the integrand function. The integral corresponding the summand $x^{-1}\left[l+1/4\right]$, transforms as follows

$$x^{-1}\left[\int\limits_{-\infty}^{+\infty}\exp(-xs^2)ds\right]\left[\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!}{4^l(l!)^2}\left[l+1/4\right]\exp\left(-x^{-1}l(l+1/2)\right)\right].$$

Substitution of this expression in Eq.(8) taking into account the value of the Poisson integral results in the formula (4).

Since the series (4) is alternating in sign then the remainder of the series does not exceed the first summand among rejected ones. Hence, the estimation (5) is valid. ■

C o r o l l a r y. *The following estimation takes place*

$$|g(x) - g_{N-1}(x)| \ < \ \frac{3}{2e^2}N^{-5/2}\,. \tag{9}$$

□ Let us estimate the remainder of the series (4). For this purpose, on the basis Eq.(4), we write down the density $g(x)$ in the form

$$g(x) \ = \ \sqrt{\frac{2}{\pi}}\sum_{l=0}^{\infty}(-1)^l\,a_l h_l(x)\,,$$

where

$$a_l = \frac{(2l)!}{4^l(l!)^2}\,(l+1/4)\,, \quad h_l(x) = x^{-3/2}\,\exp\left(-\frac{(l+1/4)^2}{x}\right),.$$

Also, we may find maximums on x of the functions $h_N(x)$, $n = 1, 2, 3, \ldots$. Equating to zero the derivative on x of this function

$$h_N'(x) = x^{-2}h_N(x)\left[(N+1/4)^2 - 3x/2\right] = 0\,,$$

we find the solution x_* of this equation. It is the point of the maximum of the function $h_N(x)$ being unique for each N,

$$x_* = \frac{2}{3}(N+1/4)^2\,,$$

$$h_N(x_*) = \left(\frac{3}{2e}\right)^{3/2} (N+1/4)^{-3}\,.$$

Hence, the estimation of the Nth remainder is

$$|g(x) - g_{N-1}(x)| \le \sqrt{\frac{2}{\pi}} a_N h_N(x_*) \;=\; \frac{3}{2}\sqrt{\frac{3}{\pi e}}\left(\frac{(2N)!}{4^N (N!)^2}\right)(N+1/4)^{-2}\,.$$

Further, we estimate the coefficient a_N having done more transparent the obtained estimation. It is made by the following way

$$a_N \;=\; \frac{(2N-1)!!}{2^N\, N!} \;=\; \prod_{l=1}^{N}\left(\frac{2l-1}{2l}\right) \;=\; \prod_{l=1}^{N}\left(1-\frac{1}{2l}\right) \;=$$

$$=\; \exp\left[\sum_{l=1}^{N} \ln(1-(2l)^{-1})\right] \;< \exp\left[-\frac{1}{2}\sum_{l=1}^{N} l^{-1}\right] \;< \exp\left[-\frac{1}{2}(1+\ln\, N)\right] \;=$$

$$=\; \frac{e^{-1/2}}{\sqrt{N}}\,,$$

in view of validity of inequalities $\ln(1-x) < -x$ at $x > 0$ and

$$\sum_{l=1}^{N}\frac{1}{l} \;>\; 1 + \int_1^N \frac{dx}{x} = 1 + \ln\, N\,.$$

Since $\sqrt{3/\pi} < 1$ then Eq.(9) takes place. ■

3. Now, we estimate the approximation accuracy of probabilities $\Pr\{J_T[w] > c\}$ which are obtained on the basis of functions $g_N(\cdot)$, $N = 1, 2, \ldots$. Since $f(x) = T^{-2}g(T^{-2}x)$ then

$$\Pr\{J_T[w] > c\} \;=\; 1 - T^{-2}\int_0^c g\left(T^{-2}x\right) dx \;\equiv\; 1 - R(c)\,,$$

where

$$R_N(c) \;=\; \int_0^{c/T^2} g_{N-1}(x) dx\,.$$

Designating the righthand side of the inequality (5) by $Q_N(x)$, we have $|g(x) - g_{N-1}(x)| \leq Q_N(x)$. Further, we determine the function

$$P_N(c) \;\equiv\; 1 \;-\; R_N(c)\,, \quad R_N(c) \;=\; \int\limits_0^{c/T^2} g_{N-1}(x)dx\,.$$

Our problem is the reception of the top estimation of the deviation $|\Pr\{J_T[w] > c\} - P_N(c)|$. From the inequality (5), it follows

$$-Q_N(x) \leq g(x) - g_{N-1}(x) \leq Q_N(x)\,.$$

Integrating between limits 0 and c/T^2, we obtain

$$-\int\limits_0^{c/T^2} Q_N(x)dx \leq R(c) - \int\limits_0^{c/T^2} g_{N-1}(x)dx \leq \int\limits_0^{c/T^2} Q_N(x)dx\,.$$

Hence,

$$|R(c) - R_N(c)| \;=\; \left| \int\limits_0^{c/T^2} \left(g(x) - g_{N-1}(x)\right) dx \right| \;\leq\; \int\limits_0^{c/T^2} Q_N(x)dx\,.$$

This gives the desired estimation

$$|\Pr\{J_T[w] > c\} - P_N(c)| \;=\; |R(c) - R_N(c)| \;\leq\; \int\limits_0^{c/T^2} Q_N(x)dx\,. \tag{10}$$

At last, we calculate the integral in the righthand side. Due to this, the estimation (10) becomes obvious

$$\int\limits_0^{c/T^2} Q_N(x)dx \;=\; a_N\sqrt{\frac{2}{\pi}} \int\limits_0^{c/T^2} \exp\left(-\frac{(N+1/4)^2}{x}\right) \frac{dx}{x^{3/2}}\,.$$

Replacement of the integration variable $y = x^{-1/2}$, $dy = -dx/2x^{3/2}$ results in the formula

$$\int\limits_0^{c/T^2} Q_N(x)dx \;=\; \sqrt{\frac{8}{\pi}} \frac{a_N}{N+1/4} \int\limits_{\frac{T(N+1/4)}{c^{1/2}}}^{\infty} e^{-y^2}dy \;=\; \frac{\sqrt{2}a_N}{N+1/4}\,\mathrm{Erfc}\left[\frac{T(N+1/4)}{c^{1/2}}\right]\,.$$

From here, using $\mathrm{Erfc}(x) \leq 1$, we find the following estimation being uniform on parameters c and T

$$\int_0^{c/T^2} Q_N(x)dx \;\leq\; \sqrt{\frac{2}{eN^3}}\,.$$

More exact estimation which takes into account the order of parameters c and T is obtained by using the standard inequality $\mathrm{Erfc}(x) < (\sqrt{\pi}x)^{-1}\exp(-x^2)$,

$$\int_0^{c/T^2} Q_N(x)dx \;\leq\; \sqrt{\frac{2c}{\pi}}\frac{a_N}{T(N+1/4)^2}\exp\left(-\frac{T^2(N+1/4)^2}{c}\right) \;<$$

$$<\; \sqrt{\frac{2c}{\pi e}}\left(TN^{5/2}\right)^{-1}\exp\left(-(TN)^2/c\right)\,.$$

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